Problem A.26

Consider the matrices

$$\mathsf{A} = \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}, \qquad \mathsf{B} = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix}.$$

(a) Verify that they are diagonalizable and that they commute.

(b) Find the eigenvalues and eigenvectors of A and verify that its spectrum is degenerate.

(c) Are the eigenvectors that you found in part (b) also eigenvectors of B? If not, find the vectors that are simultaneous eigenvectors of both matrices.

Solution

Notice that A and B have all real elements and are equal to their respective transposes. This means A and B are normal matrices and hence diagonalizable.

$$\begin{aligned} \left[\mathsf{A}^\dagger, \mathsf{A}\right] &= \mathsf{A}^\dagger\mathsf{A} - \mathsf{A}\mathsf{A}^\dagger \\ &= \widetilde{\mathsf{A}^*}\mathsf{A} - \mathsf{A}\widetilde{\mathsf{A}^*} \\ &= \widetilde{\mathsf{A}}\mathsf{A} - \mathsf{A}\widetilde{\mathsf{A}} \\ &= \mathsf{A}\mathsf{A} - \mathsf{A}\mathsf{A} \\ &= 0 \\ &\Rightarrow \quad \mathsf{A} \text{ is diagonalizable.} \end{aligned} \qquad \begin{aligned} \left[\mathsf{B}^\dagger, \mathsf{B}\right] &= \mathsf{B}^\dagger\mathsf{B} - \mathsf{B}\mathsf{B}^\dagger \\ &= \widetilde{\mathsf{B}^*}\mathsf{B} - \mathsf{B}\widetilde{\mathsf{B}^*} \\ &= \widetilde{\mathsf{B}}\mathsf{B} - \mathsf{B}\widetilde{\mathsf{B}} \\ &= \mathsf{B}\mathsf{B} - \mathsf{B}\mathsf{B} \\ &= 0 \\ &\Rightarrow \quad \mathsf{B} \text{ is diagonalizable.} \end{aligned}$$

Since

$$\mathsf{AB} = \begin{pmatrix} 0 & 9 & 0 \\ 9 & -9 & 9 \\ 0 & 9 & 0 \end{pmatrix} = \mathsf{BA},$$

the matrices, A and B, commute, meaning they can be simultaneously diagonalized.
Solve the eigenvalue problem for A.

$$\mathsf{Aa} = \lambda\mathsf{a}$$

Bring $\lambda\mathsf{a}$ to the left side and combine the terms.

$$(\mathsf{A} - \lambda\mathsf{I})\mathsf{a} = 0 \tag{1}$$

Since $\mathsf{a} \neq 0$, the matrix in parentheses must be singular, that is,

$$\det(\mathsf{A} - \lambda\mathsf{I}) = 0$$

$$\begin{vmatrix} 2-\lambda & 2 & -1 \\ 2 & -1-\lambda & 2 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0.$$

Write out the determinant and solve the equation for λ.

$$(2-\lambda)\begin{pmatrix} -1-\lambda & 2 \\ 2 & 2-\lambda \end{pmatrix} - 2\begin{pmatrix} 2 & 2 \\ -1 & 2-\lambda \end{pmatrix} - 1\begin{pmatrix} 2 & -1-\lambda \\ -1 & 2 \end{pmatrix} = 0$$

$$(2-\lambda)[(-1-\lambda)(2-\lambda)-4] - 2[2(2-\lambda)+2] - 1[4-(-1-\lambda)(-1)] = 0$$

$$-27 + 9\lambda + 3\lambda^2 - \lambda^3 = 0$$

$$-(\lambda+3)(\lambda-3)^2 = 0$$

$$\lambda = \{-3, 3\}$$

Because there are only two distinct eigenvalues, $\lambda_- = -3$ and $\lambda_+ = 3$, for this 3×3 matrix, there may be one or two eigenvectors corresponding to each. Actually, A is diagonalizable, so we know there are two eigenvectors associated with one of the eigenvalues, which means the collection of eigenvalues (the spectrum) is degenerate. To find the corresponding eigenvectors, plug λ_- and λ_+ back into equation (1).

$$(\mathsf{A} - \lambda_-\mathsf{I})\mathsf{a}_- = 0 \qquad\qquad (\mathsf{A} - \lambda_+\mathsf{I})\mathsf{a}_+ = 0$$

$$\begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad\qquad \begin{pmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 5a_1 + 2a_2 - a_3 &= 0 \\ 2a_1 + 2a_2 + 2a_3 &= 0 \\ -a_1 + 2a_2 + 5a_3 &= 0 \end{aligned}\right\} \qquad\qquad \left.\begin{aligned} -a_1 + 2a_2 - a_3 &= 0 \\ 2a_1 - 4a_2 + 2a_3 &= 0 \\ -a_1 + 2a_2 - a_3 &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} a_3 &= 5a_1 + 2a_2 \\ a_1 + a_2 + a_3 &= 0 \\ -a_1 + 2a_2 + 5a_3 &= 0 \end{aligned}\right\} \qquad\qquad \left.\begin{aligned} a_1 &= 2a_2 - a_3 \\ a_1 - 2a_2 + a_3 &= 0 \\ -a_1 + 2a_2 - a_3 &= 0 \end{aligned}\right\}$$

$$a_1 + a_2 + (5a_1 + 2a_2) = 0 \qquad\qquad a_1 = 2a_2 - a_3$$

$$6a_1 + 3a_2 = 0 \qquad\qquad a_1 = 2a_2 - a_3$$

$$\begin{aligned} -2a_1 &= a_2 \\ a_3 = 5a_1 + 2(-2a_1) &= a_1 \end{aligned} \qquad\qquad a_1 = 2a_2 - a_3$$

$$\mathsf{a}_- = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ -2a_1 \\ a_1 \end{pmatrix} \qquad\qquad \mathsf{a}_+ = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 2a_2 - a_3 \\ a_2 \\ a_3 \end{pmatrix}$$

Therefore, the eigenvectors corresponding to $\lambda_- = -3$ and $\lambda_+ = 3$ are respectively

$$\mathsf{a}_- = a_1\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathsf{a}_+ = a_2\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + a_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

where a_1, a_2, and a_3 are arbitrary (due to the fact that the eigenvalue problem is homogeneous).

Because there are two eigenvectors associated with one eigenvalue, using the three eigenvectors of A as the columns of S^{-1} won't necessarily lead to a similarity matrix S that will simultaneously diagonalize A and B. In order to determine S^{-1}, begin by finding the eigenvalues of B.

$$\mathsf{Bb} = \mu\mathsf{b}$$

Bring $\mu\mathsf{b}$ to the left side and combine the terms.

$$(\mathsf{B} - \mu\mathsf{I})\mathsf{b} = \mathsf{0}$$

Since $\mathsf{b} \neq \mathsf{0}$, the matrix in parentheses must be singular, that is,

$$\det(\mathsf{B} - \mu\mathsf{I}) = 0$$

$$\begin{vmatrix} 2-\mu & -1 & 2 \\ -1 & 5-\mu & -1 \\ 2 & -1 & 2-\mu \end{vmatrix} = 0$$

$$(2-\mu)\begin{pmatrix} 5-\mu & -1 \\ -1 & 2-\mu \end{pmatrix} + 1\begin{pmatrix} -1 & -1 \\ 2 & 2-\mu \end{pmatrix} + 2\begin{pmatrix} -1 & 5-\mu \\ 2 & -1 \end{pmatrix} = 0$$

$$(2-\mu)[(5-\mu)(2-\mu) - 1] + 1[-(2-\mu) + 2] + 2[1 - 2(5-\mu)] = 0$$

$$-18\mu + 9\mu^2 - \mu^3 = 0$$

$$-\mu(\mu - 3)(\mu - 6) = 0$$

$$\mu = \{0, 3, 6\}.$$

Now check to see if the eigenvectors of A are also eigenvectors of B.

$$\mathsf{Ba}_- = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ -2a_1 \\ a_1 \end{pmatrix} = \begin{pmatrix} 6a_1 \\ -12a_1 \\ 6a_1 \end{pmatrix} = 6\begin{pmatrix} a_1 \\ -2a_1 \\ a_1 \end{pmatrix} = 6\mathsf{a}_-$$

$$\mathsf{Ba}_{+1} = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2a_2 \\ a_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3a_2 \\ 3a_2 \\ 3a_2 \end{pmatrix} = 3\begin{pmatrix} a_2 \\ a_2 \\ a_2 \end{pmatrix}$$

$$\mathsf{Ba}_{+2} = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix}\begin{pmatrix} -a_3 \\ 0 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} -a_3 \\ 0 \\ a_3 \end{pmatrix} = 0\mathsf{a}_{+2}$$

a_{+1} is not an eigenvector of B, so replace it with a linear combination of the eigenvectors of A.

$$\begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix}\begin{pmatrix} a_1 + 2a_2 - a_3 \\ -2a_1 + a_2 \\ a_1 + a_3 \end{pmatrix} = \begin{pmatrix} 6a_1 + 3a_2 \\ -12a_1 + 3a_2 \\ 6a_1 + 3a_2 \end{pmatrix} = 3\begin{pmatrix} 2a_1 + a_2 \\ -4a_1 + a_2 \\ 2a_1 + a_2 \end{pmatrix}$$

For this column matrix to be an eigenvector of B, the following system must be satisfied.

$$\left.\begin{aligned} a_1 + 2a_2 - a_3 &= 2a_1 + a_2 \\ -2a_1 + a_2 &= -4a_1 + a_2 \\ a_1 + a_3 &= 2a_1 + a_2 \end{aligned}\right\} \quad \rightarrow \quad \begin{aligned} a_1 &= 0 \\ a_3 &= a_2 \end{aligned}$$

The replacement for a_{+1} is then

$$\begin{pmatrix} a_1+2a_2-a_3 \\ -2a_1+a_2 \\ a_1+a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ a_2 \\ a_2 \end{pmatrix} = a_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Therefore, the vectors that are simultaneous eigenvectors of A and B are

$$C_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad C_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad C_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

where C_1, C_2, and C_3 are arbitrary constants. In order to diagonalize A and B simultaneously, let S^{-1} be the 3×3 matrix whose columns are these vectors with the constants set to 1 for simplicity.

$$\mathsf{S}^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Determine S by finding the inverse of S^{-1}.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & -2 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & -2 & 2 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right)$$

Consequently,

$$\mathsf{S} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

Compute SAS^{-1} and verify that A is diagonalizable.

$$\begin{aligned}
\mathsf{SAS}^{-1} &= \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -3 & 3 & -3 \\ 6 & 3 & 0 \\ -3 & 3 & 3 \end{pmatrix} \\
&= \begin{pmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}
\end{aligned}$$

Compute SBS^{-1} and verify that B is diagonalizable.

$$\begin{aligned}
\mathsf{SBS}^{-1} &= \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 6 & 3 & 0 \\ -12 & 3 & 0 \\ 6 & 3 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$